1 An Unmeasurable Set

Proof. My textbook says that the following is a standard counterexample to all measurable sets. We may define an equivalence relation (Lemma I) such that

\[ x \approx y \iff \text{if } x - y \text{ is rational} \]

and the equivalence relation partitions \( \mathbb{R} \) into disjoint non-empty equivalence classes (Remark I). There are uncountable many equivalence classes as they map to the irrationals. Using the Axiom of Choice,\(^1\)\(^2\) we can construct a set \( A \) which contains an element in \([0, 1)\) from every equivalence class (Lemma II). Now we construct more sets using an enumeration \((x_n)_{n=1}^{\infty}\) of the rationals in \((-1, 1)\) by taking

\[ A_n = A + x_n \]

and these sets are pairwise disjoint (Lemma III). We can bound the unions of \( A_n \) with

\[ (0, 1) \subset \bigcup_{n=1}^{\infty} A_n \subset (-1, 2) \]

and with this union we can show that \( A \) is not measurable by contradiction. Falsely assume \( A \) is measurable. Then \( \forall n : m(A_n) = m(A + x_n) = m(A) \). We have

\[
m(0, 1) \leq m \left( \bigcup_{n=1}^{\infty} A_n \right) \leq m(-1, 2)
\]

\[
1 \leq \sum_{n=1}^{\infty} m(A_n) = \sum_{n=1}^{\infty} m(A) \leq 3
\]

Now these bounds tell us that neither \( m(A) = 0 \) and \( m(A) < 0 \) is possible. Therefore \( A \) must not be measurable.

Lemma I. \( x \approx y \iff \text{if } x - y \text{ is rational} \) is an equivalence relation on \( \mathbb{R} \). For a relation to be an equivalence relation it must satisfy reflexive, symmetric, and transitive properties. The relation is reflexive as for any \( x, x - x = 0 \) which is rational. The relation is symmetric as \( x - y = -(y - x) \) and sign does not affect rationality. We will now prove transitivity. If \( x \approx y \) and \( y \approx z \) then \( x - y = q_0 \) and \( y - z = q_1 \) for some rationals \( q_0, q_1 \). Now \((x - y) + (y - z) = q_0 + q_2 = x - z \). Since \( q_0 + q_1 \) is rational, then \( x \approx z \), so the relation is transitive. We can conclude that the relation is an equivalence relation.

Lemma II. The set \( \mathcal{T} \cap [0, 1) \) always contains some element, for any equivalence class \( \mathcal{T} \). Every equivalence class is non-empty so there is some real \( r \). Let \( \lfloor r \rfloor \) denote the largest integer in \((r - 1, r]\), so that

\[
r \approx r - \lfloor r \rfloor \quad \text{and} \quad r - \lfloor r \rfloor \in [0, 1)
\]

as \( r - \lfloor r \rfloor \) always has the same rationality as \( r \). We can conclude that \( r - \lfloor r \rfloor \in \mathcal{T} \cap [0, 1) \).

Lemma III. Since each irrational in \( r \in A \) was in a separate equivalence class to begin with, adding a rational to each item does not change any item’s equivalence class. Equivalence classes are disjoint, so we only need to consider each equivalence class. As \( r + x_n \) is unique for all \( n \), \( A + x_n \) is disjoint from all the other sets.

Lemma IV. The infinite union of \( A_n \) contains all elements from \((0, 1)\). For any irrational \( r \in (0, 1) \), there is an irrational in \( r_0 \in A \) where \( r_0 - r \in \mathbb{Q} \), as \( A \) contains every irrational form shifted into \([0, 1)\). Then \( r - r_0 = x_n \) for some \( n \), and \( r_0 \in A_n \). The right inclusion is simple. Since \( A \subset [0, 1] \), and the rationals \( x_n \)

\[ \text{https://en.wikipedia.org/wiki/Axiom_of_choice} \]

\[ \text{The Axiom of Choice informally states that for any collection of non-empty bins (even uncountable ones), it is possible to construct a set containing an element from each bin.} \]
range from $(-1, 1)$, all numbers in any $A_n$ range from $(-1, 2)$.

Remark I. Intuitively, the equivalence classes of $x \approx y$ are cosets $\tau = r + \mathbb{Q}$ which contain all the numbers a rational distance from $r$. For any rational $q$, $\tau = q + \mathbb{Q} = \mathbb{Q} = \emptyset$. For any irrational $r$, $\tau = r + \mathbb{Q}$. If $x = r + q_0 \in r + \mathbb{Q}$, $x \approx y$ if and only if $y \in r + \mathbb{Q}$. Equivalence classes partition the set they are defined on into disjoint non-empty sets, and the proof can be found in some abstract algebra textbook.