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1 An Unmeasurable Set

Proof. My textbook says that the following is a standard counterexample to all measurable sets. We may define an equivalence relation (*Lemma I*) such that

$$x \approx y \iff \text{if } x - y \text{ is rational}$$

and the equivalence relation partitions $\mathbb{R}$ into disjoint non-empty equivalence classes (*Remark I*). There are uncountable many equivalence classes as they map to the irrationals. Using the Axiom of Choice,[1,2] we can construct a set A which contains an element in $[0, 1)$ from every equivalence class (Lemma II). Now we construct more sets using an enumeration $(x_n)_{n=1}^{\infty}$ of the rationals in $(-1, 1)$ by taking

$$A_n = A + x_n$$

and these sets are pairwise disjoint (*Lemma III*). We can bound the unions of A_n with

$$(0, 1) \subset \bigcup_{n=1}^{\infty} A_n \subset (-1, 2)$$

and with this union we can show that A is not measurable by contradiction. Falsely assume A is measurable. Then $\forall n : m(A_n) = m(A + x_n) = m(A)$. We have

$$m(0,1) \leq m\left(\bigcup_{n=1}^{\infty} A_n\right) \leq m(-1,2)$$
$$1 \leq \sum_{n=1}^{\infty} m\left(A_n\right) = \sum_{n=1}^{\infty} m\left(A\right) \leq 3$$

Now these bounds tell us that neither $m(A) = 0$ and $m(A) < 0$ is possible. Therefore A must not be measurable.

Lemma I. $x \approx y \iff$ if $x - y$ is rational is an equivalence relation on $\mathbb{R}$. For a relation to be an equivalence relation it must satisfy reflexive, symmetric, and transitive properties. The relation is reflexive as for any x, $x - x = 0$ which is rational. The relation is symmetric as $x - y = -(y - x)$ and sign does not affect rationality. We will now prove transitivity. If $x \approx y$ and $y \approx z$ then $x - y = q_0$ and $y - z = q_1$ for some rationals q_0, q_1. Now $(x - y) + (y - z) = q_1 + q_2 = x - z$. Since $q_0 + q_1$ is rational, then $x \approx z$, so the relation is transitive. We can conclude that the relation is an equivalence relation.

Lemma II. The set $\overline{r} \cap [0, 1)$ always contains some element, for any equivalence class $\overline{r}$. Every equivalence class is non-empty so there is some real r. Let $[r]$ denote the largest integer in $(r - 1, r]$, so that

$$r \approx r - [r] \quad \text{ and } r - [r] \in [0, 1)$$

as $r - [r]$ always has the same rationality as r. We can conclude that $r - [r] \in \overline{r} \cap [0, 1)$.

Lemma III. Since each irrational in $r \in A$ was in a separate equivalence class to begin with, adding a rational to each item does not change any item's equivalence class. Equivalence classes are disjoint, so we only need to consider each equivalence class. As $r + x_n$ is unique for all n, $A + x_n$ is disjoint from all the other sets.

Lemma IV. The infinite union of A_n contains all elements from $(0, 1)$. For any irrational $r \in (0, 1)$, there is an irrational in $r_0 \in A$ where $r_0 - r \in \mathbb{Q}$, as A contains every irrational form shifted into $[0, 1)$. Then $r - r_0 = x_n$ for some n, and $r_0 \in A_n$. The right inclusion is simple. Since $A \subset [0, 1]$, and the rationals x_n

[1] `https://en.wikipedia.org/wiki/Axiom_of_choice`

[2] The Axiom of Choice informally states that for any collection of non-empty bins (even uncountable ones), it is possible to construct a set containing an element from each bin.

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range from $(-1, 1)$, all numbers in any A_n range from $(-1, 2)$.

Remark I. Intuitively, the equivalence classes of $x \approx y$ are cosets $\overline{r} = r + \mathbb{Q}$ which contain all the numbers a rational distance from r. For any rational q, $\overline{q} = q + \mathbb{Q} = \mathbb{Q} = \overline{0}$. For any irrational r, $\overline{r} = r + \mathbb{Q}$. If $x = r + q_0 \in r + \mathbb{Q}$, $x \approx y$ if and only if $y \in r + \mathbb{Q}$. Equivalence classes partition the set they are defined on into disjoint non-empty sets, and the proof can be found in some abstract algebra textbook.